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# AN EFFICIENT ALGORITHM FOR SOLVING THE FRACTIONAL DIRAC DIFFERENTIAL OPERATOR

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**Abstract.** In this manuscript, we offer a systematic calculative algorithm to solve the one–dimensional fractional Dirac operator. The method of solution is according to utilizing the series solution to convert the governing system of fractional differential equations into a linear system of algebraic equations. We obtain the corresponding polynomial characteristic equations for some types of boundary conditions relying on the polynomial expansion and integral technique. Therewith, the eigenvalues can be discovered simultaneously from the multi–roots. Finally, we use some numerical examples to show that this method includes to demonstrating the validity and applicability of the technique.

**Keywords**: Dirac operator, Fractional differential equation, Eigenvalue, Eigenfunction.
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## 1 Introduction

Consider the system of differential equation as bellows

$$\ell[y(x)] := By'(x) + \Omega(x)y(x) = \lambda y(x), \quad x \in (a,b), \tag{1}$$

with

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \Omega(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{bmatrix}, \ y(x) = (y_1(x), y_2(x))^T,$$

subject to the boundary conditions

$$U(y) := \sum_{i=1}^{2} r_{1i} y_i(a) = 0,$$
$$V(y) := \sum_{i=1}^{2} r_{2i} y_i(b) = 0. \tag{2}$$

Throughout this paper, $p_{ij}(x)$ in $L(a,b)$, and $r_{ij}$ for $i,j = 0,1$ are real and $\lambda$ is the spectral parameter.

In the case where $p_{12}(x) = p_{21}(x) = 0$, $p_{22}(x) = V(x) + m$, and $p_{11}(x) = V(x) - m$, $V(x)$ is the potential function and $m$ is the mass of particle. Eq. (1) is named a *one–dimensional stationary Dirac system* in relativistic quantum theory.

Using a smooth and orthogonal transformation of a two dimensional space, we get

$$\Omega(x) = \begin{bmatrix} p(x) & 0 \\ 0 & q(x) \end{bmatrix} \text{ or } \Omega(x) = \begin{bmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{bmatrix}.$$

These are called the *canonical forms of Dirac operator* (see Sargsjan et al. (2012)).

The fundamental and immense results about Dirac operator were given in Sargsjan et al. (2012). In the seminal paper, Amirov motivated by the direct (studied of the asymptotic forms of eigenvalues and eigenfunctions). Inverse problems for Dirac operator with discontinuities inside an interval were studied in Amirov (2005). In addition, direct or inverse spectral problems for Dirac operator were extensively investigated in Guo et al. (2015); Mochizuki & Trooshin (2001, 2002); Yang (2011) and the references therein. Unfortunately, it is difficult to achieve the exact eigenvalues and eigenfunctions for most cases.

In this manuscript, we offer a systematic calculative algorithm to obtain the eigenvalues and eigenfunctions for the one dimensional fractional Dirac operator. During the last few years, the fractional Sturm–Liouville problems and the fractional Dirac operator have been of interest to various researchers. It should be considered that since analytical solutions for this problem is an extremely laborious task, many numerical algorithms for fractional Sturm-Liouville problems have been developed to seek approximate solutions. For instance, the homotopy analysis method and the Adomian decomposition method for Abbasbandy & Shirzadi (2010); Al-Mdallal (2009), the iterative approximation method, Neamaty et al. (2009), the variational methods Klimek & Agrawal (2013), theoretical and computational perspectives on the eigenvalues of fractional fourth-order, Al-Mdallal et al. (2017) and an efficient power series Hajji et al. (2014); Huang et al. (2013), have been used.

## 2 Preliminaries

Analogous to problem (1)–(2), we take the system of fractional differential equation as bellows

$$\ell^\alpha[y(x)] := BD^\alpha y(x) + \Omega(x)y(x) = \lambda y(x), \quad x \in (a,b), \tag{3}$$

with

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \Omega(x) = \begin{bmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{bmatrix},\ \text{or } \Omega(x) = \begin{bmatrix} p(x) & 0 \\ 0 & r(x) \end{bmatrix},$$

$y(x) = [y_1(x), y_2(x)]^T$ , and $\frac{1}{2} < \alpha \leq 1$, subject to the boundary conditions (2). In this paper, $p(x)$ and $q(x)$ are real functions in $L(a,b)$ and $\lambda$ is the spectral parameter. Here $D^\alpha$ signifies the fractional differential operator of order $\alpha$. We introduce the characteristic function for the operator $\ell^\alpha$ of the form

$$\Delta(\lambda) := V(\varphi(\lambda)),$$

where $\varphi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))^T$ is the solution of (3) satisfying in $U(\varphi(\lambda)) = 0$ in (2). The characteristic function $\Delta(\lambda)$ does not depend on $x$. The function $\Delta(\lambda)$ has an at most countable set of zeroes $\lambda_n$, for $n \in \{\mathbb{Z} - \{0\}\}$

The notation $D^\alpha$ for any $\alpha \in \mathbb{R}^+$ signifies the left sided Caputo fractional derivative defined by

$$D^\alpha y(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} y^{(m)}(t)dt, \quad x > 0, \tag{4}$$

where $m = [\alpha]$. We offer some fundamental information about fractional calculus theory that will be anxiously utilized in this paper. At the first, we bring the definition of the Riemann–Liouville fractional integral operator of order $\alpha$.

**Definition 1.** *The left sided Riemann–Liouville fractional integral operator of order $\alpha$ is signified by*

$$J^{\alpha}y(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}y(t)dt, \tag{5}$$

*where $y\in L_1[0,T]$, $x\le T$, and $\alpha\in\mathbb{R}^+$.*

Some useful properties of the operator $J^{\alpha}$ are summarized in the following Lemma. (Miller & Ross, 1993; Podlubny, 1998).

**Lemma 1.** *Let $\alpha$, $\beta$, $x>0$, and $\gamma>-1$. Then*

*(i)* $J^{\alpha}J^{\beta}=J^{\beta}J^{\alpha}=J^{\alpha+\beta}$,

*(ii)* $J^{\alpha}x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}x^{\gamma+\alpha}$.

We consider that the left side of Caputo fractional derivative (4) is originally defined through the left sided Riemann–Liouville fractional integral (5), (see Mainardi & Gorenflo (2008)). So we have

$$D^{\alpha}y(x)=J^{m-\alpha}y^{(m)}(x),\quad x>0.$$

**Lemma 2.** *For $\alpha\in\mathbb{R}^+$, $m=[\alpha]$ and $y\in L_1[0,T]$, we have*

*(i)* $D^{\alpha}J^{\alpha}y(x)=y(x)$,

*(ii)* $J^{\alpha}D^{\alpha}y(x)=y(x)-\sum\limits_{k=0}^{m-1}y^{(k)}(0+)\frac{x^k}{k!}$,

*(iii)* $D^{\alpha}x^{r}=\begin{cases}\frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}x^{r-\alpha}, & \text{for } [\alpha]<r;\\ 0, & \text{for } [\alpha]\ge r.\end{cases}$

For Case (iii), we use the notation $\Gamma_{\alpha,r}=\frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}$.

## 3 Formulas and method

Particularly, the problem becomes solving a set of the fractional differential equations (3) with the boundary conditions (2). Unfortunately, it is difficult to obtain exact eigenvalues of Eqs. (3) and (2). Accordingly, the numerical methods must be proposed to solve such an eigenvalue problem. In this section, avoiding solving Eq. (3) directly, by using the similar methods of Hajji et al. (2014); Huang et al. (2013), we introduce a simple method to determine the eigenvalues and corresponding the eigenfunctions of the Dirac differential operator. For this purpose, we expand the solution $y(x)$ of Eq. (3) in the following polynomial form as:

$$y(x)=\begin{bmatrix} y_1(x)\\ y_2(x)\end{bmatrix}=\begin{bmatrix}\sum\limits_{i=0}^{N}c_{1i}x^i+R_{1N}(x)\\ \sum\limits_{i=0}^{N}c_{2i}x^i+R_{2N}(x)\end{bmatrix}, \tag{6}$$

where $c_{1i}$ and $c_{2i}$ are unknown coefficients, $R_{1N}$ and $R_{2N}$ are the truncation error, and $N$ is a certain positive integer which is selected large enough such that the rest has a negligible error. We further suppose that our solution can be approximated by

$$\begin{bmatrix} y_1(x)\\ y_2(x)\end{bmatrix}\approx\begin{bmatrix}\sum\limits_{i=0}^{N}c_{1i}x^i\\ \sum\limits_{i=0}^{N}c_{2i}x^i\end{bmatrix}. \tag{7}$$

Ultimately, the main idea is to obtain a homogeneous system of equation in the unknowns $c_{1i}$ and $c_{2i}$, $i = 0, 1, \dots N$, the roots of whose characteristic equation constitute the eigenvalues of the problem. First, from the boundary conditions (2) and using (7) we obtain the following equations

$$\sum_{i=0}^{2N+1} (f_{m,i} - \lambda k_{m,i}) c_i = 0, \quad m = 0, 1, \tag{8}$$

where

$$f_{0,i} = \sum_{m=0}^{N} r_{11} a^m + \sum_{m=N+1}^{2N+1} r_{12} a^{m-N-1}, \quad k_{0,i} = 0, \quad i = 1, 2, \dots, N, \tag{9}$$

and

$$f_{1,i} = \sum_{m=0}^{N} r_{21} b^m + \sum_{m=N+1}^{2N+1} r_{22} b^{m-N-1}, \quad k_{1,i} = 0, \qquad i = 1, 2, \dots, N, \tag{10}$$

with $c_i = c_{1i}$ for $i = 0, 1, \dots, N$ and $c_i = c_{2(i-N-1)}$ for $i = N+1, N+2, \dots, 2N+1$.

Next, substituting (7) in (3), using the linearity property and Lemma 2, we get

$$\sum_{i=0}^{2N+1} c_i \left( D_2^i \Gamma_{\alpha, i-(N+1)} x^{i-\alpha-N-1} + D_1^i p(x) x^i + D_2^i q(x) x^{i-N-1} \right) - \lambda \sum_{i=0}^{2N+1} c_i D_1^i x^i = 0, \tag{11}$$

$$-\sum_{i=0}^{2N+1} c_i \left( D_1^i \Gamma_{\alpha,i} x^{i-\alpha} + D_1^i q(x) x^i - D_2^i p(x) x^{i-N-1} \right) - \lambda \sum_{i=0}^{2N+1} c_i D_2^i x^{i-N-1} = 0, \tag{12}$$

where

$$D_1^i = \begin{cases} 1, & \text{for } i = 0, 1, \dots, N, \\ 0, & \text{for } i = N+1, N+2, \dots, 2N+1, \end{cases}$$

and

$$D_2^i = \begin{cases} 0, & \text{for } i = 0, 1, \dots, N, \\ 1, & \text{for } i = N+1, N+2, \dots, 2N+1. \end{cases}$$

We note that to obtain other linear algebraic equations of unknown coefficients $c_i$, we multiply both sides of (11) by $x^l (l = 0, 1, \dots, N)$ and then integrate with respect to $a \le x \le b$. We get

$$\sum_{i=0}^{2N+1} (f_{ji} - \lambda k_{ji}) c_i = 0, \quad j = 2, 3, \dots, 2N+1, \tag{13}$$

where

$$\begin{aligned} f_{ji} &= \int_a^b \left( D_2^i \Gamma_{\alpha, i-(N+1)} x^{i-\alpha-N-1} + D_1^i p(x) x^{i+j-2} + D_2^i q(x) x^{i+j-N-3} \right) dx, \\ k_{ji} &= \frac{D_1^i (b^{i+j-1} - a^{i+j-1})}{i+j-1}, \quad j = 2, 3, \dots, N+1 \end{aligned} \tag{14}$$

and

$$\begin{aligned} f_{ji} &= -\int_a^b \left( D_1^i \Gamma_{\alpha,i} x^{i-\alpha} + D_1^i q(x) x^{i+j-N-2} - D_2^i p(x) x^{i+j-2N-3} \right) dx, \\ k_{ji} &= \frac{D_2^i (b^{i+j-2N-2} - a^{i+j-2N-2})}{i+j-2N-2}, \quad j = N+2, N+3, \dots, 2N+1. \end{aligned} \tag{15}$$

Therefore, from Eqs. (9), (10), and (13) we get the following linear system of $2N+2$ equations.

$$\sum_{i=0}^{2N+1} (f_{ji} - \lambda k_{ji}) c_i = 0, \quad j = 0, 1, \dots, 2N+1. \tag{16}$$

For simplicity, the system (16) can be written in matrix form

$$(\mathbf{F} - \lambda \mathbf{K})\mathbf{C} = 0, \tag{17}$$

where $\mathbf{F}$ and $\mathbf{K}$ are square $(2N+2) \times (2N+2)$ matrices with $F_{mi} = f_{mi}$ and $K_{mi} = k_{mi}$, and $\mathbf{C} = (c_0, c_1, \ldots, c_{2N+1})^t$. To obtain a nontrivial solution of the system of equations, the determinant of the coefficient matrix of the system must be vanish; then we get a characteristic function in eigenvalues $\lambda$:

$$\det(\mathbf{F} - \lambda \mathbf{K}) = 0, \tag{18}$$

such that $\det(\mathbf{F} - \lambda \mathbf{K})$ is a polynomial of degree $2N$ in $\lambda$. The eigenvalues of the original problem would be those that satisfy (18). In our simulations, we solve (18) via the Matlab built-in function *solve()*. We consider that even if the matrices $\mathbf{F}$ and $\mathbf{K}$ have high condition numbers as $N$ increases, by using the command *solve()* solving the Eq. (18) was stable.

In this part, by using our numerical method, two algorithms for calculating eigenvalues and eigenfunctions of the problem (3), (2) are presented.

**Algorithm 1** Find the eigenvalues of problem (3) and (2).

1. Define the symbol $x$ and $\lambda$.
2. Define $a,\ b,\ r_{kj},\ p(x),\ q(x)$.
3. Define $N$ the degree of power series.
4. Construct the matrices $\mathbf{F}$ and $\mathbf{K}$ using (9)-(10) and (14)-(15).
5. Define the symbolic matrix $\mathbf{A} = \mathbf{F} - \lambda \mathbf{K}$.
6. Calculate $P = \det(\mathbf{A})$.
7. Solve for the eigenvalues of $\mathbf{A}$ by calling the Matlab function *solve*$(P)$.

and

**Algorithm 2** Find the eigenfunction for a particular eigenvalue $\lambda$ of problem (3) and (2).

1. Substitute the value of the particular $\lambda$ in the matrix $\mathbf{A}$ by using the Algorithm 1.
2. Use the function *null*$(\mathbf{A})$ to fined a basis for the null space of $\mathbf{A}$.
3. Pike the vector from *null*$(\mathbf{A})$ and construct the eigenfunction, $y_1(x)$ and $y_2(x)$ by (7).

## 4 Numerical results

In this section, we consider four examples to demonstrate the performance and efficiency of the present algorithms.

**Example 1.** *Consider the fractional Dirac operator* (3) *with* $p(x) = 0$, $q(x) = 0$, *and the boundary conditions*

$$y_1(0) = 0, \qquad y_2'(1) = 0.$$

*The exact eigenvalues of this problem for* $\alpha = 1$ *are* $\lambda_n = n\pi$ *for* $n \in \mathbb{Z} - \{0\}$.

**Example 2.** *In the second example, we consider the fractional Dirac operator* (3) *with*

$$\Omega(x) = \begin{bmatrix} \cos(x) & \sin(x) \\ \sin(x) & -\cos(x) \end{bmatrix},$$

*and the boundary conditions*

$$y_1(0) - y_2(0) = 0, \qquad y_1(1) = 0.$$

**Table 1:** The numerical values of the first 8 eigenvalues in example 1 for $n$ = 11, 19 and $\alpha$ = 1, 0.99, 0.95, 0.9, 0.85

| $k$ | $\lambda_k$ Exact | $\Lambda_k^{11}$ $\alpha=0.99$ | $\Lambda_k^{11}$ $\alpha=0.95$ | $\Lambda_k^{11}$ $\alpha=0.9$ | $\Lambda_k^{11}$ $\alpha=0.85$ |
|---|---|---|---|---|---|
| −4 | −12.56637061436 | −12.22234238732 | −11.03076268655 | −9.841682460921 | −8.869123916119 |
| −3 | −9.424777960769 | −9.201059087738 | −8.376483433255 | −7.476339866930 | −6.641716584301 |
| −2 | −6.283185307180 | −6.154614333469 | −5.682460358770 | −5.185075296835 | −4.800928560367 |
| −1 | −3.141592653590 | −3.089990347841 | −2.897576583713 | −2.685799192265 | −2.501228925925 |
| 1 | 3.14159265358980 | 3.0899903478411 | 2.8975765837132 | 2.68579919226455 | 2.50122892592547 |
| 2 | 6.2831853071796 | 6.1546143334693 | 5.68246035876596 | 5.18507529683501 | 4.80092856036662 |
| 3 | 9.4247779607694 | 9.2010590877382 | 8.37648343325457 | 7.4763398669309 | 6.64171658430135 |
| 4 | 12.566370614359 | 12.222342387322 | 11.0307626865466 | 9.84168246092064 | 8.86912391611947 |

| $k$ | $\Lambda_k^{19}$ $\alpha=1$ | $\Lambda_k^{19}$ $\alpha=0.99$ | $\Lambda_k^{19}$ $\alpha=0.95$ | $\Lambda_k^{19}$ $\alpha=0.9$ | $\Lambda_k^{19}$ $\alpha=0.85$ |
|---|---|---|---|---|---|
| −4 | −12.566370510565 | −12.23863028894 | −11.04145718815 | −9.735285830407 | −8.376408124096 |
| −3 | −9.4247779607694 | −9.201427078535 | −8.380636090392 | −7.522190778198 | −6.928898000870 |
| −2 | −6.2831853071799 | −6.154236207975 | −5.679505635301 | −5.163974257757 | −4.707108104399 |
| −1 | −3.1415926535898 | −3.090169006520 | −2.898845110691 | −2.693182597114 | −2.526688942854 |
| 1 | 3.14159265358979 | 3.09016900651954 | 2.89884511069107 | 2.69318259711445 | 2.52668894285415 |
| 2 | 6.28318530717988 | 6.15423620797408 | 5.67950563530144 | 5.16397425775732 | 4.70710810439959 |
| 3 | 9.42477796076938 | 9.20142707853545 | 8.38063609039211 | 7.52219077819788 | 6.92889800087042 |
| 4 | 12.5663705105647 | 12.2386302889351 | 11.0414571881458 | 9.73528583040716 | 8.37640812409643 |

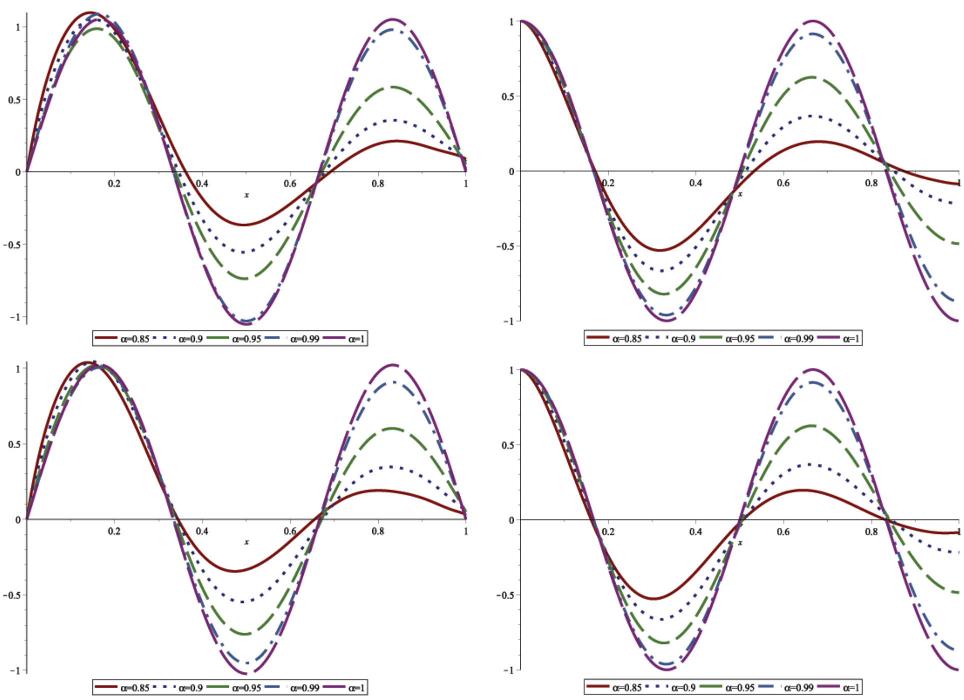


**Figure 1:** The approximation of normalized eigenfunctions $y_1(x)$ and $y_2(x)$ corresponding to the third eigenvalue, $\lambda_3$, for $q(x) = 0$, $p(x) = 0$, respectively, with $\alpha = 1$, 0.99, 0.95, 0.9, 0.85, and $N = 11, 19$.

**Table 2:** The numerical values of the first 8 eigenvalues in example 2 with $n = 15$ and $\alpha = 0.99$, 0.95, 0.9, 0.85

| $k$ | $\Lambda_k^{15}$, $\alpha = 0.99$ | $\Lambda_k^{15}$, $\alpha = 0.95$ | $\Lambda_k^{15}$, $\alpha = 0.9$ | $\Lambda_k^{15}$, $\alpha = 0.85$ |
|---|---|---|---|---|
| −4 | −11.53091608556 | −10.48260252024 | −9.345286264097 | −8.297907385235 |
| −3 | −8.505595747845 | −7.842076666211 | −7.14744891543 | −6.624795862106 |
| −2 | −5.479526081088 | −5.140802137687 | −4.771401283344 | −4.449382123145 |
| −1 | −2.498565754061 | −2.441615139070 | −2.388926090730 | −2.360100758632 |
| 1 | 0.589692631064 | 0.584485990671 | 0.579101263549 | 0.574898376762 |
| 2 | 3.990903168854 | 3.812884892353 | 3.633570638236 | 3.508879000627 |
| 3 | 6.994267827067 | 6.491202655561 | 5.939341168717 | 5.444524115711 |
| 4 | 10.02111643532 | 9.176260539002 | 8.298611533246 | 7.673755973309 |

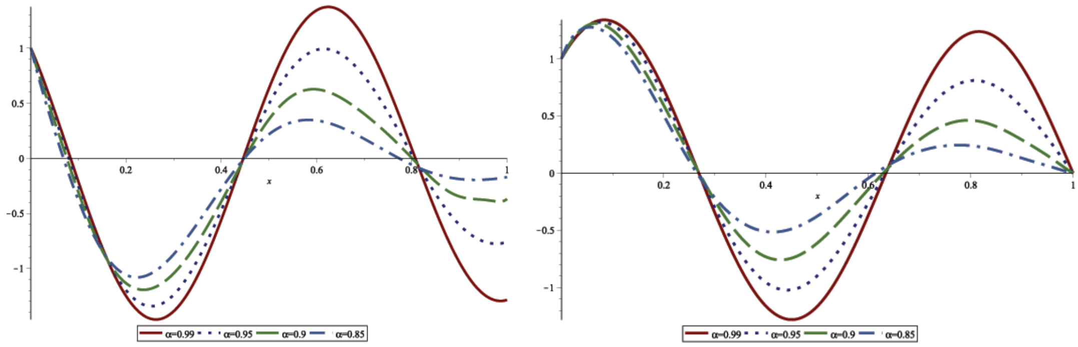


**Figure 2:** The approximation of normalized eigenfunctions $y_1(x)$ and $y_2(x)$ corresponding to the third eigenvalue, $\lambda_3$, for $p(x) = \cos x,\ q(x) = \sin x$, respectively, with $\alpha = 0.99,\ 0.95,\ 0.9,$ and $0.85$.

**Example 3.** *In this example, we consider the fractional Dirac operator* (3) *with*

$$\Omega(x) = \begin{bmatrix} \frac{1}{\sqrt{x}} & 10x^2 \\ 10x^2 & -\frac{1}{\sqrt{x}} \end{bmatrix}$$

*and the following boundary conditions*

$$y_1(0) - y_2(0) = 0, \qquad y_1(1) + y_2(1) = 0.$$

**Table 3:** The numerical values of the first 8 eigenvalues in example 3with $n = 15$ and $\alpha = 1,\ 0.99,\ 0.95,\ 0.9,\ 0.86$

| $k$ | $\Lambda_k^{15}$ $\alpha = 1$ | $\Lambda_k^{15}$ $\alpha = 0.99$ | $\Lambda_k^{15}$ $\alpha = 0.95$ | $\Lambda_k^{15}$ $\alpha = 0.9$ | $\Lambda_k^{15}$ $\alpha = 0.86$ |
|---|---|---|---|---|---|
| $-4$ | $-11.871897673$ | $-11.6492806348$ | $-10.8479711071$ | $-9.976420387365$ | $-9.0919526985$ |
| $-3$ | $-9.3373107889$ | $-9.22894807651$ | $-8.85128754083$ | $-8.521989961032$ | $-8.54859191698$ |
| $-2$ | $-7.0314864460$ | $-6.96832105342$ | $-6.70727682434$ | $-6.355830122885$ | $-6.02693254009$ |
| $-1$ | $-3.8559666749$ | $-3.85318428779$ | $-3.84568052340$ | $-3.867294991354$ | $-3.89976332466$ |
| 1 | 0.66011127820 | 0.66198719874 | 0.68626634996 | 0.665114480134 | 0.67541365368 |
| 2 | 5.82968639698 | 5.78182040708 | 5.61433319966 | 5.443655707715 | 5.35779608094 |
| 3 | 8.84585623080 | 8.71429400824 | 8.23298783818 | 7.694289408712 | 7.28946960416 |
| 4 | 11.7556789187 | 11.5270509664 | 10.7062754158 | 9.865968891682 | 9.38379130362 |

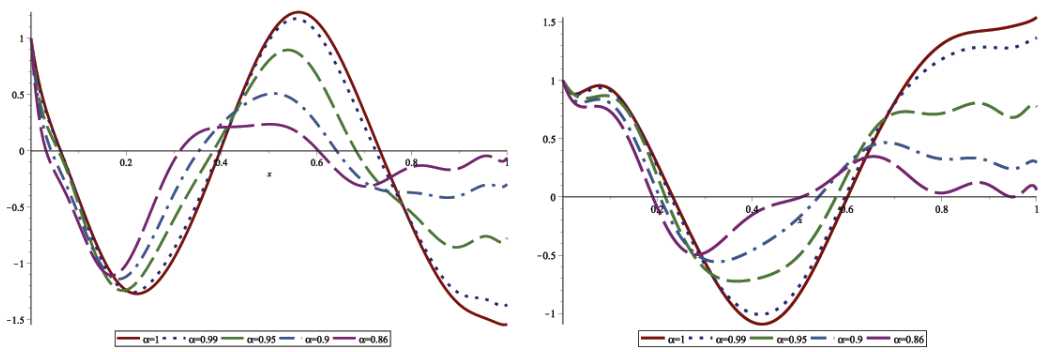


**Figure 3:** The approximation of normalized eigenfunctions $y_1(x)$ and $y_2(x)$ corresponding to the third eigenvalue, $\lambda_3$, for $p(x) = \frac{1}{\sqrt{x}},\ q(x) = 10x^2$, respectively, with $\alpha = 1,\ 0.99,\ 0.95,\ 0.9,\ 0.86$.

**Example 4.** *Finally, consider the canonical fractional Dirac operator* (3) *with*

$$\Omega(x) = \begin{bmatrix} 10x^2 & 0 \\ 0 & \sin(x) \end{bmatrix},$$

*and the following boundary conditions*

$$y_1(0) - y_2(0) = 0, \qquad y_1(1) = 0.$$

**Table 4:** The numerical values of the first 8 eigenvalues in example 4 with $n = 15$ and $\alpha = 1,\, 0.99,\, 0.95,\, 0.85$

| $k$ | $\Lambda_k^{15}\alpha = 1$ | $\Lambda_k^{15}\alpha = 0.99$ | $\Lambda_k^{15}\alpha = 0.95$ | $\Lambda_k^{15}\alpha = 0.85$ |
|---|---|---|---|---|
| $-4$ | $-10.03843189632$ | $-9.75432549753488$ | $-8.7108636126896$ | $-6.6077023372311$ |
| $-3$ | $-6.9479842950844$ | $-6.76689395275573$ | $-6.1017131822017$ | $-4.8292970478217$ |
| $-2$ | $-3.919390862295$ | $-3.82818487569058$ | $-3.4907348228755$ | $-2.8211347634262$ |
| $-1$ | $-0.972657349657$ | $-0.94506048394982$ | $-0.8408365257368$ | $-0.6237781969919$ |
| 1 | 1.325007640634 | 1.3233073119681 | 1.317445441621 | 1.309420032123 |
| 2 | 5.658086505721 | 5.6094584962064 | 5.436225561058 | 5.200525809800 |
| 3 | 9.093851755115 | 8.9561769301407 | 8.437378436515 | 7.214427102554 |
| 4 | 12.22572796588 | 12.037556095065 | 11.20888832556 | 10.06514516570 |

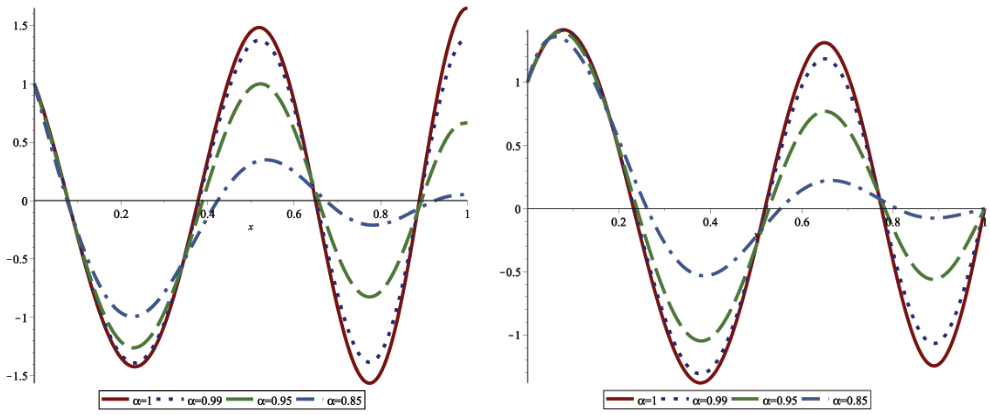


**Figure 4:** The approximation of normalized eigenfunctions $y_1(x)$ and $y_2(x)$ corresponding to the fourth eigenvalue, $\lambda_4$, for $p(x) = 10x^2$, $q(x) = \sin x$, respectively, with $\alpha = 1,\, 0.99,\, 0.95,\, 0.85$.

## 5 Conclusion

In this paper, the numerical solution of the fractional Dirac operator and Robin boundary conditions investigated. For this purpose, an efficient computational algorithm for solving the one–dimensional fractional Dirac operator presented. The corresponding polynomial characteristic equations for some types of boundary conditions according to the polynomial expansion and integral method was obtained. Finally, some numerical examples utilized to show that this method includes to illustrating the validity and applicability of the technique.

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